Remark on my former paper "On an extension of Löwner's theorem".

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Theorem. *Let* $w=f(z)$ *be regular and* $|f(z)|<1$ *in* $|z|<1$, $f(0)=0$. *Then* $\lim\limits_{r\to 1} f(re^{i\theta})=f(e^{i\theta})$ *exists for almost all* $e^{i\theta}$ *on* $|z|=1$. *Let* E_0 *be the set of* $e^{i\theta}$, *such that* $|f(e^{i\theta})|=1$, *then* E_0 *is a Borel set. Let* E *be any Borel sub-set of* E_0 *and* E^* *be its image on* $|w|=1$. *Then* E^* *is an analytic set, so that it is measurable and*

$$mE \leqq mE^*.$$

If $0<mE_0<2\pi$, $mE>0$, *then* $mE<mE^*$.

In my former paper[1], I assumed that $E=E_0$, but the general case can be proved without any modification of the proof. Though the proof is the same as my former one, we shall reproduce the proof for the sake of completeness. We remark that Ohtsuka[2] also obtained the same result. Löwner[3] proved the case, where E is an arc on $|z|=1$ and Kawakami[4] the case, where $f(z)$ is schlicht in $|z|<1$ and $E=E_0$ and Kametani and Ugaheri[5] proved that for any sub-set E of E_0, $m_iE\leqq m_eE^*$, where m_i, m_e denote the inner and the outer measure.

Proof. Since for a fixed $r(0\leqq r<1)$, $f(re^{i\theta})$ is a continuous function of θ, by Hahn's theorem[6], the set A of $e^{i\theta}$, such that $\lim\limits_{r\to 1} f(re^{i\theta})=f(e^{i\theta})$ exists is an $E_{\sigma\delta}$-set, so that $f(e^{i\theta})$ is a Borel function, defined on a Borel set A. Since E_0 is the set of $e^{i\theta}$, such that $|f(e^{i\theta})|=1$, E_0 is a Borel set. Let E be any Borel sub-set of E_0 and put for $e^{i\theta}\in E$, $f(e^{i\theta})=e^{i\psi(\theta)}$. Consider on the (θ, ψ)-plane, the set M of points $(\theta, \psi(\theta))$, $e^{i\theta}\in E$, then M is a Borel set. Since E^* is the projection of a Borel set M on the ψ-axis, E^* is an analytic set, so that it is measurable.

We may assume that $mE>0$ and put

$$u(z)=\int_E \frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta, \quad u^*(w)=\int_{E^*}\frac{1-|w|^2}{|w-e^{i\psi}|^2}d\psi\,, \tag{1}$$

1) M. Tsuji: On an extension of Löwner's theorem. Proc. Imp. Acad. Tokyo. **18** (1941).

2) M. Ohtsuka: Dirichlet problems on Riemann surfaces and conformal mappings. Nagoya Math. Journ. **3** (1951).

3) K. Löwner: Untersuchungen über schichte konforme Abbildungen des Einheitskreises. I. Math. Ann. **89** (1923).

4) Y. Kawakami: On an extension of Löwer's lemma. Jap. Journ. Math **17** (1941).

5) S. Kametani and T. Ugaheri: A remark on Kawakami's extension of Löwner's lemma. Proc. Imp. Acad. Tokyo. **18** (1941).

6) Hausdorf. Mengenlehre., p. 271.

$$v(z)=u^*(f(z))-u(z)\,. \tag{2}$$

Let O be an open set on $|w|=1$, which contains E^* and we define $u_0^*(w)$ similarly as $u^*(w)$ with O instead of E^* and put

$$v_0(z)=u_0^*(f(z))-u(z)\,. \tag{3}$$

Then $v_0(z)$ is a bounded harmonic function in $|z|<1$, and $\lim\limits_{r\to1} v(re^{i\theta})=0$ almost everywhere on E and $\geqq 0$ almost everywhere on the complement E' of E, so that $v_0(z)\geqq 0$ in $|z|<1$, hence if we make $mO\to mE^*$, then we have $v(z)\geqq 0$ in $|z|<1$, so that $v(0)=u^*(0)-u(0)\geqq 0$, or

$$mE\leqq mE^*. \tag{4}$$

Next suppose that $0<mE_0<2\pi$, $mE>0$, then $0<mE\leqq mE^*$, hence

$$u^*(w)>0 \text{ in } |w|<1\,. \tag{5}$$

If $mE=mE^*$, then $v(0)=0$, so that $v(z)\equiv 0$, or $u^*(f(z))\equiv u(z)$.
Since $0<mE_0<2\pi$, there exists a point $e^{i\theta}$ in the complement E_0' of E_0, such that

$$\lim_{r\to1} u(re^{i\theta})=0, \quad \lim_{r\to1} f(re^{i\theta})=w_0, \quad |w_0|<1\,.$$

Hence $u^*(w_0)=0$, which contradicts (5), so that $mE<mE^*$.

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